

# Counting Votes for Aggregating Judgments

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## ABSTRACT

The aim of judgment aggregation is to make collective decisions based on the judgments of individual agents. Some rationality conditions governing the expected behavior of the aggregation function must be considered. However, impossibility theorems show that designing an aggregation function satisfying all desirable properties is not feasible. While some rationality conditions are very natural ones, other ones are more disputable. We show that this is the case of the systematicity condition that prevents from electing issues with more votes than others. We rather promote a neutrality and a swap optimality condition. Swap optimality ensures that among two possible results, the one with the best support (number of votes) is chosen. We propose a new family of judgment aggregation methods based on the support (number of votes) that receives each issue.

## Categories and Subject Descriptors

[Computing methodologies]: Artificial intelligence : Knowledge representation and reasoning

## General Terms

Theory

## Keywords

Social choice theory, Judgment Aggregation

## 1. INTRODUCTION

Judgment aggregation aims at making collective decisions on a set of issues from the opinions (judgments) expressed by individuals on these issues. These issues are usually not logically independent. The output of such an aggregation process is one or several social judgment sets, reflecting the opinion of the group of agents on the issues. Judgment aggregation is a very active research topic in Social Choice Theory, Political Philosophy and Artificial Intelligence [18, 15, 12, 2, 8, 16, 11, 9, 3]. Much work about judgment aggregation has been devoted to the study of rationality properties for aggregation methods, and the main results are impossibility theorems, showing that no judgment aggregation method satisfies all expected properties [15, 13, 16, 9].

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	1	2	3	4	5	6	7	maj
$\varphi_1$	1	1	1	1	1	0	0	1
$\varphi_2$	1	1	1	0	0	1	1	1
$\varphi_3$	0	0	0	1	1	1	1	1

Table 1: A doctrinal paradox

Let us illustrate the difficulty raised up when aggregating judgments by considering a simple example. Consider the three propositional formulae  $\varphi_1 = a$ ,  $\varphi_2 = b$ ,  $\varphi_3 = \neg a \vee \neg b$ : it is not possible to accept  $\varphi_1$ ,  $\varphi_2$  and  $\varphi_3$  together since they are jointly inconsistent. Table 1 reports the opinions of each individual from a group of seven people  $\{1, 2, 3, 4, 5, 6, 7\}$  on these three formulae (1 means "accept" and 0 "reject"). For example, agent 1's judgment is that  $a$  is true, that  $b$  is also true, and that  $\neg a \vee \neg b$  is not true. Each individual opinion is only constrained by a rationality condition of consistency: for any agent, the set of judgments she reports is supposed to be consistent. A natural method to make a collective decision is to use a majority vote: if a majority of agents accepts an issue, then this issue is accepted by the group. Using this method, since a majority of individuals are for  $\varphi_1$ , for  $\varphi_2$  and for  $\varphi_3$ , the decision made by the majority should be  $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ , which is not consistent. Examples of this kind are called doctrinal paradoxes (see [16] for more details).

While this example illustrates that simple majority vote does not work as an admissible judgment aggregation (JA) method, a key issue is to determine alternative methods to do the job. Echoing impossibility theorems in voting theory, impossibility theorems for judgment aggregation state that there is no method satisfying the full set of expected rationality properties [15, 13, 16, 9]. As a consequence, some rationality conditions must be given up.

Many impossibility theorems consider a property referred to as systematicity, which basically states that the collective judgment on each issue is the same function of individual judgments on that issue, whatever the votes on the other issues. This property may be seen as a way to ensure some form of strategy-proofness: the collective decision cannot be changed by adding or removing some issues of the agenda (this property is related to the Independence of Irrelevant Alternatives in voting theory [1]). However, systematicity prevents from considering judgment aggregation as an optimization process: an issue is accepted or rejected independently of the votes on the other issues. Thus, adhering to systematicity may prevent from selecting issues getting more votes than others, because comparison between issues is not allowed. In our opinion, this behaviour is not desirable.

Contrastingly, many existing judgments aggregation methods do not satisfy the neutrality condition, that we consider as a first class requirement. Neutrality intuitively states that all the issues have to be considered on an equal basis, without any priority between

them. The premise-based method [18, 2, 6], the conclusion-based method [18, 2, 20], the sequential priority rule [12] violate neutrality, by requiring additional information to make a decision (such as a set of identified premises). Of course, when some additional information is available, such as a distinction between premises and conclusions, or some priority between the formulae, then this extra information has to be taken into account and neutrality must be given up. However, in the remaining case, neutrality is a way to avoid some arbitrariness to take place in the aggregation process. Distance-based methods [19] and more generally the methods based on minimization recently proposed in [11] satisfy it.

In this paper, we first criticize systematicity in presence of neutrality. Chapman [3] also criticizes systematicity, because this property prevents from considering the logical interactions between issues to solve the conflicts. Our work is different from Chapman's because we consider the case where no additional information is available, so neutrality has to be satisfied. We provide both a natural example highlighting the limits of systematicity, and an impossibility theorem showing that it conflicts with other expected rationality properties. Then we define and study judgment aggregation methods which select consistent sets of issues as social judgment sets, based on the support (number of votes) of their elements. As a matter of illustration, on the previous example, one can easily observe that each of  $\varphi_1$  and  $\varphi_2$  receives 5 (out of 7) votes, whereas  $\varphi_3$  receives only 4 votes. While the three issues are accepted by a majority of individuals, the supports of  $\varphi_1$  and of  $\varphi_2$  are strictly greater than the one of  $\varphi_3$ . This can be considered as a sufficient evidence for preferring the judgment set  $\{\varphi_1, \varphi_2\}$  (and so  $\neg\varphi_3$ ), to the judgment sets  $\{\varphi_2, \varphi_3\}$  and  $\{\varphi_1, \varphi_3\}$ .

Quite surprisingly, only few JA methods take such support information into account; the most notable exception is Porello-Endriss' method [21], equivalent to Lang et al.'s ranked agenda method [11], which is the instantiation to JA of Tideman's ranked pairs method [23, 26]. This is rather astonishing since, if one interprets these aggregations from an epistemic point of view<sup>1</sup>, then the more votes an issue gets, then the more likely it is (see Section 5 for a more formal discussion).

We introduce the family of Support-based Aggregation Correspondences (SAC). This family gathers methods for which the selection of an issue in a social judgment set depends only on its logical interactions with the other issues and on the level of support it gets from the individuals. We specifically focus on the subset of SAC consisting of ranked majority methods. Such methods select consistent judgment sets based on the number of votes received by their elements. We study the rationality properties offered by these methods both in the general case and for some specific methods. We also discuss the significance of some of the ranked majority methods using truth tracking arguments.

The rest of the paper is organized as follows. Section 2 first gives some formal preliminaries. Then some rationality properties for aggregation methods are recalled and discussed in Section 3. SAC and ranked majority methods are defined in Section 4. In Section 5 several ranked majority methods are characterized thanks to truth tracking arguments. Finally, Section 6 concludes the paper. For space reasons, only the proofs of the main results are reported.

## 2. FORMAL PRELIMINARIES

An *agenda* is a finite, non-empty and totally ordered set of non-trivial (i.e., non-contradictory and non-tautological) propositional formulae  $X = \{\varphi_1, \dots, \varphi_m\}$ .

A *judgment* on a formula  $\varphi_k$  of  $X$  is an element of  $D = \{1, 0, \star\}$ ,

<sup>1</sup>Under usual reliability and independence assumptions [4, 14, 7].

where 1 means that  $\varphi_k$  is supported, 0 that  $\neg\varphi_k$  is supported,  $\star$  that neither  $\varphi_k$  nor  $\neg\varphi_k$  are supported. A *judgment set* on  $X$  is a mapping  $\gamma$  from  $X$  to  $D$ , also viewed as the set of formulae  $\{\varphi_k \mid \varphi_k \in X \text{ and } \gamma(\varphi_k) = 1\} \cup \{\neg\varphi_k \mid \varphi_k \in X \text{ and } \gamma(\varphi_k) = 0\}$ . For each  $\varphi_k$  of  $X$ ,  $\gamma$  is supposed to satisfy  $\gamma(\neg\varphi_k) = \neg\gamma(\varphi_k)$ , where  $\neg\gamma$  is given by  $\neg\gamma(\varphi_k) = \star$  iff  $\gamma(\varphi_k) = \star$ ,  $\neg\gamma(\varphi_k) = 1$  iff  $\gamma(\varphi_k) = 0$ , and  $\neg\gamma(\varphi_k) = 0$  iff  $\gamma(\varphi_k) = 1$ .

Judgment sets are often asked to be consistent and resolute: A judgment set  $\gamma$  on  $X$  is *consistent* iff  $\bigwedge_{\{\varphi_k \in X \mid \gamma(\varphi_k)=1\}} \varphi_k \wedge \bigwedge_{\{\varphi_k \in X \mid \gamma(\varphi_k)=0\}} \neg\varphi_k$  is consistent. It is *resolute* iff  $\forall \varphi_k \in X$ ,  $\gamma(\varphi_k) = 0$  or  $\gamma(\varphi_k) = 1$ .

Aggregating judgments consists in associating a collective judgment set with a profile of individual judgment sets: A *profile*  $P = (\gamma_1, \dots, \gamma_n)$  on  $X$  is a vector of judgment sets on  $X$ . It is *consistent* (resp. *resolute*) when each judgment set in it is consistent (resp. resolute).

For each agenda  $X$ , a *judgment aggregation method*  $\delta$  associates with a profile  $P$  on  $X$  a non-empty set  $\delta(P)$  of judgment sets  $\gamma_P$  on  $X$ . When  $\delta(P)$  is a singleton for each  $P$ , the judgment aggregation method is called a (deterministic) judgment aggregation rule, and it is called a judgment aggregation correspondence otherwise [11].

## 3. RATIONALITY ISSUES

What are the properties judgment aggregation methods should satisfy? Many intuitions are captured by the following properties<sup>2</sup>. The first one states that no specific condition must be imposed on the input profile, but consistency:

**Universal domain.** The domain of  $\delta$  is the set of all consistent profiles.

This property is often relaxed [15] to:

**R-universal domain.** The domain of  $\delta$  is the set of all profiles which are consistent and resolute.

Some properties also state that the result should be consistent and resolute:

**Collective rationality.** For any profile  $P$  in the domain of  $\delta$ ,  $\delta(P)$  is a set of consistent collective judgment sets.

**Collective completeness.**<sup>3</sup> For any profile  $P$  in the domain of  $\delta$ ,  $\delta(P)$  is a set of resolute collective judgment sets.

It is usually expected that agents play symmetric roles:

**Anonymity.** For any two profiles  $P = (\gamma_1, \dots, \gamma_n)$  and  $P' = (\gamma'_1, \dots, \gamma'_n)$  in the domain of  $\delta$  which are permutations one another, we have  $\delta(P) = \delta(P')$ .

Systematicity states that issues receiving the same support must be treated in the same way:

**Systematicity.** For any two profiles  $P = (\gamma_1, \dots, \gamma_n)$  and  $P' = (\gamma'_1, \dots, \gamma'_n)$  in the domain of  $\delta$ , and any two propositions  $\varphi_k$  and  $\varphi_l$  of  $X$ , such that  $\gamma_i(\varphi_k) = \gamma'_i(\varphi_l) \forall i$ , if  $\gamma_P(\varphi_k) = x$  for all  $\gamma_P \in \delta(P)$ , then  $\gamma_{P'}(\varphi_l) = x$  for all  $\gamma_{P'} \in \delta(P')$ .

Unanimity states that if all agents agree on the judgment on an issue, then the collective judgment on this issue must be the unanimous one:

**Unanimity.** For any  $\varphi_k \in X$ , if  $\gamma_i(\varphi_k) = x$  with  $x \in \{0, 1\}$ ,  $\forall \gamma_i \in P$ , then for every  $\gamma_P \in \delta(P)$ , we have  $\gamma_P(\varphi_k) = x$ .

<sup>2</sup>Most of these properties are usually stated for judgment aggregation rules and for resolute profiles. We translate these properties in the more general setting of judgment aggregation correspondences and without the resoluteness assumption. See [22] for a similar translation.

<sup>3</sup>Sometimes also called (**collective resoluteness**).

Note that Unanimity is not stated when all agents vote  $\star$  for a formula. In this case it does not seem desirable to force  $\star$  for the collective decision, since the value of this formula may be imposed by the decisions on other logically related formulas.

In order to define the next property which expresses a form of compliance to majority, we need a few preliminary definitions: let  $g_P^+$ ,  $g_P^-$  and  $g_P$  be the majority counting functions from  $X$  to  $\mathbb{Z}$  given by  $g_P^+(\varphi_k) = |\{\gamma_i \in P \mid \gamma_i(\varphi_k) = 1\}|$  and  $g_P^-(\varphi_k) = |\{\gamma_i \in P \mid \gamma_i(\varphi_k) = 0\}|$ , and  $g_P(\varphi_k) = g_P^+(\varphi_k) - g_P^-(\varphi_k)$ . These functions can be straightforwardly extended to mappings from  $X \cup \{\neg\varphi_k \mid \varphi_k \in X\}$  to  $\mathbb{Z}$  so that for any  $\varphi_k \in X$ , we have  $g_P^+(\neg\varphi_k) = g_P^-(\varphi_k)$ ,  $g_P^-(\neg\varphi_k) = g_P^+(\varphi_k)$ , and  $g_P(\neg\varphi_k) = -g_P(\varphi_k)$ .

We can now define formally the majoritarian aggregation rule  $\delta^{maj}$ , which is the judgment aggregation rule we considered in the example given in the introduction.

**DEFINITION 1.** *The majoritarian aggregation rule  $\delta^{maj}$  is defined as follows. For any agenda  $X$  and any profile  $P$  on  $X$ , we have  $\delta^{maj}(P) = \{\gamma_P^{maj}\}$ , where for any issue  $\varphi_k \in X$ :*

$$\gamma_P^{maj}(\varphi_k) = \begin{cases} 1 & \text{if } g_P(\varphi_k) > 0 \\ 0 & \text{if } g_P(\varphi_k) < 0 \\ \star & \text{if } g_P(\varphi_k) = 0 \end{cases}$$

We are now in position to define the majority preservation property:

**Majority preservation.** If  $\gamma_P^{maj}$  is consistent and resolute, then  $\delta(P) = \{\delta^{maj}(P)\}$ .

Majority preservation<sup>4</sup> [22] is a very natural property, stating that if the simple majority vote on each issue leads to a consistent judgment set, then the judgment aggregation correspondence must contain this set, and no other set. The idea is to stick to the result furnished by a simple majority vote when no doctrinal paradox occurs.

Some additional logical properties seem also reasonable for judgment aggregation correspondences. Thus, when all the information available is given by the input profile, it is expected that issues play symmetric roles:

**Neutrality.** If  $X = \{\varphi_1, \dots, \varphi_m\}$  and  $X' = \{\varphi'_1, \dots, \varphi'_m\}$  are two agendas such that there exists a permutation  $\sigma$  over  $\{1, \dots, m\}$  satisfying  $\varphi_k = \varphi'_{\sigma(k)}$  for every  $k \in \{1, \dots, m\}$ , then for any profiles  $P = (\gamma_1, \dots, \gamma_n)$  on  $X$  and  $P' = (\gamma'_1, \dots, \gamma'_n)$  on  $X'$  such that for every  $i \in \{1, \dots, n\}$ , for every  $k \in \{1, \dots, m\}$ ,  $\gamma_i(\varphi_k) = \gamma'_i(\varphi'_{\sigma(k)})$ , we have  $\delta(P) = \delta(P')$ .

We want to stress that this neutrality property is different from the one usually considered in judgment aggregation, where systematicity is equivalent to the two properties of independence and "neutrality" [15]. Both properties differ even in the restricted case of resolute profiles. The usual "neutrality" property is a weakening of systematicity, and is far from the standard meaning of neutrality in vote theory, which is the one we want to capture by our property.

Finally, let us define the swap  $\gamma_P|_S$  of a collective judgment set  $\gamma_P$  with respect to a set of issues  $S \subseteq X$  as  $\gamma_P|_S(\varphi_k) = \gamma_P(\varphi_k)$  if  $\varphi_k \notin S$ , and  $\gamma_P|_S(\varphi_k) = \neg\gamma_P(\varphi_k)$  if  $\varphi_k \in S$ . We may consider the following property:

**Swap optimality.** If  $\gamma_P \in \delta(P)$ , then there are no  $\varphi_k, \varphi_l \in X$  such that  $\gamma_P|_{\{\varphi_k, \varphi_l\}}$  is consistent and  $\max(g_P^+(\neg\varphi_k), g_P^+(\neg\varphi_l)) > \max(g_P^+(\varphi_k), g_P^+(\varphi_l))$ .

Swap optimality requires that the selection of collective judgment sets depends on the number of votes received by each issue.

<sup>4</sup>Called strong majority preservation in [22].

$P$	$\varphi_1$	$\varphi_2$	$\varphi_3$	$\varphi_4$	$\varphi_5$	$\varphi_6$	$P'$	$\varphi_1$	$\varphi_2$	$\varphi_3$	$\varphi_4$	$\varphi_5$	$\varphi_6$
1	1	1	0	1	1	1	1'	0	1	1	1	1	1
2	1	1	1	0	1	1	2'	0	1	1	1	1	1
3	1	1	1	1	0	1	3'	0	1	1	1	1	1
4	1	1	1	1	1	0	4'	0	1	1	1	1	1
5	1	0	1	1	1	1	5'	1	0	0	0	0	0
6	1	0	1	1	1	1	6'	1	0	0	0	0	0

**Table 2: Profiles  $P$  and  $P'$**

So if one can consistently replace in a collective judgment set any pair of formulae of the agenda by their negations, one should not get a better number of votes for any of these two negated formulae. This property ensures that most supported formulae are chosen from a profile, when it is possible to do so. R-swap optimality is the relaxation of swap optimality where only resolute profiles  $P$  are considered. In this case, the expression of the swap optimality condition is simpler:

**R-swap optimality.** If  $P$  is resolute and  $\gamma_P \in \delta(P)$ , then there are no  $\varphi_k, \varphi_l \in X$  such that  $\gamma_P|_{\{\varphi_k, \varphi_l\}}$  is consistent and  $g_P^+(\varphi_k) + g_P^+(\varphi_l) < n$ , where  $n$  is the number of agents in the profile.

This simpler expression allows for a better understanding of what swap optimality means. Overall, R-swap optimality aims at keeping the most supported issues in the selected result: if  $g_P^+(\varphi_k) + g_P^+(\varphi_l) < n$  and if it is possible to swap  $\varphi_k$  to  $\neg\varphi_k$  and  $\varphi_l$  to  $\neg\varphi_l$  in the profile, then the profile with the two negated formulae would receive strictly more votes than the initial one.

It turns out that all these properties are not jointly compatible. We first recall the impossibility theorem of [15] (for judgment aggregation rules only).

**PROPOSITION 1 ([15]).** *There exists no judgment aggregation rule that satisfy R-universal domain, collective rationality, collective completeness, systematicity and anonymity.*

This theorem is quite negative, but it relies on some strong assumptions. First is the resoluteness (completeness) assumptions of the individuals (R-universal domain), that can be criticized, since one cannot expect all agents to have an opinion on all possible issues; this is also the case of the collective completeness property, that is helpful for making decisions, but forces to make some choices even when there is no evidence enough to do so. Thus the collective completeness requirement imposes sometimes to discriminate further some judgment sets, using additional information not given in the input profile. As such, it conflicts with the anonymity and neutrality conditions. Suppose for instance a perfect tie (say, about a unique issue  $\varphi$  in the agenda, with 4 votes for and 4 votes against it), why and how to make a distinction between  $\varphi$  and  $\neg\varphi$ ? See [8] for criticisms on the collective completeness property.

The systematicity property is also highly criticizable. It prevents from considering judgment aggregation as an optimization process trying to achieve a best compromise, which is often expected for aggregation methods. The following example illustrates it:

**EXAMPLE 1.** *Let us consider an agenda  $X$  composed of the following six formulae:  $\varphi_1 = (\neg a \vee \neg b \vee \neg c \vee \neg d \vee \neg e)$ ,  $\varphi_2 = a$ ,  $\varphi_3 = b$ ,  $\varphi_4 = c$ ,  $\varphi_5 = d$ ,  $\varphi_6 = e$ . Let us consider the profiles  $P$  and  $P'$  on this agenda, as given by Table 2. In the (resolute) profile  $P$ , every formula has a majority of votes, so using simple majority vote all the formulae have to be selected, which would lead to an inconsistent collective judgment set. So (at least) one of the six formulae has to be rejected by the judgment aggregation correspondence. There is a unanimity for  $\varphi_1$ , so it seems*

sensible to select  $\varphi_1$  in the result. All the other formulae except  $\varphi_2$  are quasi-unanimously accepted (they get all votes but one). The least supported formula is  $\varphi_2$ , so the expected result is  $\gamma_P = \{\varphi_1, \neg\varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6\}$ .

Consider now the profile  $P'$ . Simple majority vote leads to a consistent collective judgment set  $\gamma_{P'} = \{\neg\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6\}$ , which thus appears as the expected result. So, though the individual judgments are the same ones in both profiles for  $\varphi_2$ , in the expected result for  $P \neg\varphi_2$  is selected, whereas for  $P'$   $\varphi_2$  is selected. Since  $\varphi_2$  gets the same votes pros and cons in the two profiles, no judgment aggregation method satisfying systematicity is allowed to make such a distinction.

This example illustrates clearly that the individual judgments on an issue cannot be considered independently from those for the other issues. This requirement conflicts with those supported by the other rationality conditions, leading to the following impossibility theorem:

**PROPOSITION 2.** *There is no judgment aggregation correspondence that satisfies R-universal domain, unanimity, majority preservation, R-swap optimality, and systematicity.*

**PROOF.** Consider Example 1, and a judgment aggregation correspondence satisfying universal domain, unanimity, majority preservation and swap optimality. As this judgment aggregation correspondence satisfies universal domain, the two profiles are acceptable. From the unanimity assumption,  $\varphi_1$  has to be in the result for the first profile. Since the six formulae are not jointly consistent, the negation of at least one of them has to be chosen. Using the swap optimality assumption,  $\varphi_2$  must be rejected because its support is lower than the one of  $\varphi_3, \varphi_4, \varphi_5$  or  $\varphi_6$ . For the second profile, the majority preservation assumption forces the result, as the majoritarian rule gives a consistent judgment set:  $\neg\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$  and  $\varphi_6$  is the result. This judgment aggregation correspondence selects  $\neg\varphi_2$  for the first profile,  $\varphi_2$  for the second profile, whereas the support of  $\varphi_2$  is exactly the same one in both profiles. Hence this correspondence does not satisfy systematicity.  $\square$

**COROLLARY 1.** *There is no judgment aggregation correspondence that satisfies universal domain, unanimity, majority preservation, swap optimality, and systematicity.*

We want to stress that systematicity, which is often presented as the counterpart of Independence of Irrelevant Alternatives (IIA) for voting methods (surely because Dietrich and List [5] have shown that when one wants to express Arrow's Theorem in a judgment aggregation setting, using propositions to encode the preference relation, then IIA leads to the related Independence property), actually is stronger than IIA (see [17] for a similar statement). For this reason, systematicity is set aside in the rest of the paper: we now look for judgment aggregation correspondences satisfying universal domain, collective rationality, anonymity, neutrality, majority preservation, unanimity, and swap (or R-swap) optimality. In the next section, we prove that such a correspondence exists.

## 4. SUPPORT-BASED JUDGMENT AGGREGATION

Let us now point out a general family of judgment aggregation correspondences for which the selection of consistent collective judgment sets is based only on the support obtained by their elements. To achieve this goal, let us first introduce a few definitions:

**DEFINITION 2.** *Given an agenda  $X = \{\varphi_1, \dots, \varphi_m\}$ , a potential solution  $M = \langle \alpha_1, \dots, \alpha_m \rangle$  is a vector such that each  $\alpha_k$*

*is either  $\varphi_k$  or  $\neg\varphi_k$ , and  $\bigwedge_{i=1}^m \alpha_i$  is consistent.  $M_X$  is the set of all potential solutions given  $X$ .*

Obviously, there is a direct equivalence between the potential solutions and the consistent and resolute collective judgment sets. Observe that  $M_X$  is never empty.

**DEFINITION 3.** *Given a profile  $P$  on  $X$ , the support set of  $P$  is the multiset  $ss(P) = \{\langle \varphi_i, g_P^+(\varphi_i), g_P^-(\varphi_i) \rangle \mid \varphi_i \in M\} \mid M \in M_X\}$ .*

Thus, the support set of a profile  $P$  reports the evidence pro and against every issue of  $X$ , for each potential solution given  $X$ .

**DEFINITION 4.** *A Support-based Aggregation Correspondence (SAC)  $\delta$  is a judgment aggregation correspondence such that there exists a function  $f$  such that for any consistent profile  $P$  on  $X$ ,  $\delta(P) = f(ss(P))$ .*

The definition of SAC therefore calls simply the result of the aggregation process for each agenda to be determined solely by the votes for or against each issue (no further information such as a classification of formulae into premises and conclusions is taken into account).

Clearly enough, each SAC already satisfies a number of properties of interest:

**PROPOSITION 3.** *Every SAC satisfies universal domain, collective rationality, anonymity and neutrality.*

While the majoritarian aggregation rule  $\delta^{maj}$  defined previously is not a SAC since it does not satisfy the collective rationality condition, we can easily define a SAC based on it, and more generally, on qualified majority methods (quota methods):<sup>5</sup>

**DEFINITION 5.** *Let  $q$  be any integer. The  $q$ -quota aggregation rule  $\delta^{Qq}$  is defined as follows. For any agenda  $X$ , and any profile  $P$  on  $X$ , we have  $\delta^{Qq}(P) = \{\gamma_P^{Qq}\}$ , where for any  $\varphi \in X$ :*

$$\gamma_P^{Qq}(\varphi) = \begin{cases} 1 & \text{if } g_P(\varphi) > q \\ 0 & \text{if } g_P(\varphi) < q \\ * & \text{otherwise} \end{cases}$$

*The SAC  $\delta_{sac}^{Qq}$  associated with  $\delta^{Qq}$  is defined as  $\delta_{sac}^{Qq}(P) = \{M \in M_X \mid \text{if } \gamma_P^{Qq}(\varphi) = 1, \text{ then } \varphi \in M \text{ and if } \gamma_P^{Qq}(\varphi) = 0, \text{ then } \neg\varphi \in M\} \text{ if } \gamma_P^{Qq} \text{ is consistent, and } \delta_{sac}^{Qq}(P) = M_X \text{ otherwise.}$*

The  $\delta_{sac}^{Qq}$  methods clearly suffers from a lack of accuracy whenever  $\gamma_P^{Qq}$  is inconsistent (all the potential solutions are kept in this case).

Let us now introduce a specific family of SAC which does not exhibit this drawback, the family of ranked majority methods. For such correspondences, the selection of potential solutions is based on their aggregated score:

**DEFINITION 6.** *The score vector of a potential solution  $M = \langle \alpha_1, \dots, \alpha_m \rangle$  given  $P$  is the vector of votes in favor of the issues selected by this potential solution:*

$$s(M) = (g_P^+(\alpha_1), \dots, g_P^+(\alpha_m)).$$

<sup>5</sup>  $\delta^{maj}$  is the 0-quota aggregation method  $\delta^{Q0}$ .

Note that  $s(M)$  is defined using only the "positive" support  $g_P^+$  of issues. There is no need to consider the "negative" support  $g_P^-$  as well, since the negations of formulae from  $X$  are also considered in other potential solutions.

DEFINITION 7. An aggregation function is a mapping<sup>6</sup>  $\oplus$  from  $\mathbb{R}^m$  to  $\mathbb{R}$ , which satisfies:

- If  $x_i \geq x'_i$ , then  $\oplus(x_1, \dots, x_i, \dots, x_m) \geq \oplus(x_1, \dots, x'_i, \dots, x_m)$  **(non-decreasingness)**
- $\oplus(0, \dots, 0) = 0$  **(minimality)**
- If  $\sigma$  is a permutation over  $\{1, \dots, m\}$ , then  $\oplus(x_1, \dots, x_m) = \oplus(x_{\sigma(1)}, \dots, x_{\sigma(m)})$  **(symmetry)**

Some additional properties can also be considered for the  $\oplus$  function:

- If  $x_i > x'_i$ , then  $\oplus(x_1, \dots, x_i, \dots, x_m) > \oplus(x_1, \dots, x'_i, \dots, x_m)$  **(strict increasingness)**
- If  $\oplus(x_1, \dots, x_m, z) \geq \oplus(y_1, \dots, y_m, z)$  **(decomposition)** then  $\oplus(x_1, \dots, x_m) \geq \oplus(y_1, \dots, y_m)$
- If  $\forall i, z > y_i$  then  $\oplus(z, x_1, \dots, x_m) > \oplus(y_1, \dots, y_{m+1})$  **(strict preference)**

DEFINITION 8. A ranked majority judgment aggregation method  $\delta^{RM\oplus}$  associates with every profile  $P$  on agenda  $X$  the set of potential solutions with the greatest score vector w.r.t. the  $\oplus$  aggregation function, i.e.,  $\delta^{RM\oplus}(P) = \{M \in M_X \mid \nexists M' \in M_X \text{ s.t. } \oplus(s(M')) > \oplus(s(M))\}$ .

As expected, it is easy to show that:

PROPOSITION 4. Every ranked majority judgment aggregation method is a SAC.

Some judgment aggregation methods pointed out in the literature belong to the family of ranked majority judgment aggregation methods. Thus, the maxweight subagenda rule introduced in [11] is the ranked majority method with  $\oplus = \Sigma$ . This particular method also has a distance-based characterization, since it actually selects the consistent judgment sets that minimize the Hamming distance w.r.t. the judgment profile. But other ranked majority judgment aggregation methods (in the general case, i.e., when  $\oplus \neq \Sigma$ ) lead to methods which are not distance-based, because ranked majority judgment aggregation methods consider the support of each formula (on columns), whereas the distances are computed on profiles (on lines). Another method defined in [11], namely the operator  $\delta^{RA}$ , is not a ranked majority method but the  $\delta^{RMleximax}$  (which is a ranked majority judgment aggregation method) refines  $\delta^{RA}$ . Other meaningful aggregation functions can be used to give rise to other ranked majority methods. Let us mention for instance  $max$ ,  $min$ ,  $leximax$ ,  $leximin$ <sup>7</sup>,  $\Sigma^n$  (the sum of the  $n^{th}$  powers), etc.

Let us introduce a last aggregation function, that will allow us to define an interesting ranked majority judgment aggregation method.

DEFINITION 9. Given a positive number  $q$ , let  $maj_q$  be the aggregation function given by:

$$maj_q(x_1, \dots, x_n) = |\{x_i > q \mid i \in \{1, \dots, n\}\}|$$

For any profile  $P$  on  $X$ , let us define  $Maj$  as  $maj_{\lfloor \frac{|P|}{2} \rfloor}$ .

<sup>6</sup>Strictly speaking, it is a family of mappings, one for each  $m$ .

<sup>7</sup>The standard  $leximax$  and  $leximin$  functions can be encoded numerically using Ordered Weighted Averages functions [24, 10] to comply with Definition 7.

Ranked majority methods  $\delta^{RMmaj_q}$  select the potential solutions with the highest aggregated score, where the score of a solution is the number of formulae in it which are supported by a majority (which depends on the value of  $q$ ). Interestingly,  $\delta^{RMmaj}$  differs from each of  $\delta^{RMleximax}$ ,  $\delta^{RMleximin}$ , and  $\delta^{RM\Sigma}$ :

EXAMPLE 2. Let  $X = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$  be the agenda with  $\varphi_1 = a$ ,  $\varphi_2 = b$ ,  $\varphi_3 = \{(a \vee b) \wedge c\}$  and  $\varphi_4 = (\neg a \vee \neg b) \wedge d$ , and let the profile  $P$  on  $X$  be given by the following table.

	$\varphi_1$	$\varphi_2$	$\varphi_3$	$\varphi_4$
1	1	0	1	1
2	1	0	0	1
3	1	1	0	0
4	1	1	1	0
5	1	1	0	0
6	0	1	0	1
7	0	1	1	1

Each of  $M_1$ ,  $M_2$ ,  $M_3$  below is a potential solution given  $P$ :

- $M_1 = \langle \varphi_1, \varphi_2, \varphi_3, \neg \varphi_4 \rangle$ , and  $s(M_1) = (5, 5, 3, 3)$
- $M_2 = \langle \varphi_1, \neg \varphi_2, \neg \varphi_3, \varphi_4 \rangle$ , and  $s(M_2) = (5, 2, 4, 4)$
- $M_3 = \langle \neg \varphi_1, \varphi_2, \neg \varphi_3, \varphi_4 \rangle$ , and  $s(M_3) = (2, 5, 4, 4)$

Each of  $M_2$  and  $M_3$  is strictly preferred to  $M_1$  when  $Maj$  is the aggregation function since they contain 3 formulae with a majority whereas  $M_1$  contains only 2 formulae with a majority. Such a conclusion cannot be drawn when  $leximax$ ,  $leximin$  or  $\Sigma$  are used: each of  $\delta^{RMleximax}$ ,  $\delta^{RMleximin}$ , and  $\delta^{RM\Sigma}$  selects  $M_1$ .

Let us now study the rationality conditions satisfied by ranked majority methods  $\delta_{RM\oplus}$ :

PROPOSITION 5. • Any  $\delta_{RM\oplus}$  satisfies universal domain, collective rationality, anonymity and neutrality.

- If  $\oplus$  satisfies strict increasingness, then  $\delta_{RM\oplus}$  satisfies majority preservation.
- If  $\oplus$  satisfies strict preference and decomposition, then  $\delta_{RM\oplus}$  satisfies unanimity and swap optimality.

PROOF. We give the proof for the third result, the proofs for the two first results are much easier.

• If  $\oplus$  satisfies strict preference and decomposition, then  $\gamma_{RM\oplus}$  satisfies unanimity.

Let  $\varphi$  be a formula supported by all agents,  $n$  be the number of agents. There is a potential solution  $M_u$  which contains all the formulae supported by all agents (since agents are supposed to report consistent judgment sets). Consider another potential solution  $M$ , and suppose that  $M$  does not contain  $\varphi$ . Let  $(n, \dots, n, x_1, \dots, x_k)$  be the score of  $M_u$  and  $(n, \dots, n, y_1, \dots, y_{k+l})$  be the score of  $M$ , with  $l \geq 1$  (we know that  $s(M)$  contains at least one  $n$  less than  $s(M_u)$ , as  $\varphi$  is in  $M_u$  and not in  $M$ ). As  $\forall i, n > y_i$ ,  $\oplus(n, x_1, \dots, x_k) > \oplus(y_1, \dots, y_{k+l})$  (strict preference). Thanks to decomposition, we obtain  $\oplus(n, \dots, n, x_1, \dots, x_k) > \oplus(n, \dots, n, y_1, \dots, y_{k+l})$  (in the remaining case, if  $\oplus(n, \dots, n, x_1, \dots, x_k) \leq \oplus(n, \dots, n, y_1, \dots, y_{k+l})$ , with decomposition we get  $\oplus(n, x_1, \dots, x_k) \leq \oplus(y_1, \dots, y_{k+l})$ : contradiction).

With non-decreasingness, we get  $\oplus(n, \dots, n, y_1, \dots, y_{k+l}) \leq \oplus(n, \dots, n, y_1, \dots, y_{k+l})$ , because  $y_{k+2} \leq n, \dots, y_{k+l} \leq n$ . Then the result holds:  $\oplus(n, \dots, n, x_1, \dots, x_k) > \oplus(n, \dots, n, y_1, \dots, y_{k+l})$ . Equivalently,  $\oplus(s(M_u)) > \oplus(s(M))$ , and  $\varphi$  is chosen.

• If  $\oplus$  satisfies strict preference and composition, then  $\gamma_{RM\oplus}$  satisfies swap optimality.

Suppose  $\gamma_P \in \delta(P)$ , and  $\varphi_k, \varphi_l \in X$  such that  $\gamma_P|_{\{\varphi_k, \varphi_l\}}$  is consistent and  $\max(s(\neg \varphi_k), s(\neg \varphi_l)) > \max(s(\varphi_k), s(\varphi_l))$ . Then as  $\oplus$  satisfies strict preference,  $\oplus(s(\neg \varphi_k), s(\neg \varphi_l)) > \oplus(s(\varphi_k), s(\varphi_l))$ . Then with decomposition  $\oplus_{i \in X, i \neq k, l}(s(\varphi_i), s(\neg$

$\varphi_k), s(\neg\varphi_l)) > \oplus_{\varphi_i \in X, i \neq k, l} (s(\varphi_i), s(\varphi_k), s(\varphi_l))$ :  $\gamma_P|_{\{\varphi_k, \varphi_l\}}$  is selected and not  $\gamma_P$ , contradiction. There is no  $\varphi_k, \varphi_l \in X$  such that  $\gamma_P|_{\{\varphi_k, \varphi_l\}}$  is consistent and  $\max(s(\neg\varphi_k), s(\neg\varphi_l)) > \max(s(\varphi_k), s(\varphi_l))$ : swap optimality is satisfied.  $\square$

As expected, the systematicity property is not satisfied in general by  $\delta_{RM_{\oplus}}$  (this is an easy consequence of Theorem 1 and Proposition 5). However, the remaining properties can be jointly satisfied, leading thus to a "possibility theorem":

**PROPOSITION 6.**  $\delta_{RM_{leximax}}$  satisfies universal domain, majority preservation, anonymity, unanimity, collective rationality, neutrality and swap optimality.

The next proposition illustrates the importance of the completeness condition on the input profile:

**PROPOSITION 7.** If  $P$  is a resolute profile, then  $\delta_{RM_{leximax}}(P) = \delta_{RM_{leximin}}(P)$ .

**PROOF.** For space reasons, we give only a sketch of the proof. The first point is to prove that the lexicographic order between two vectors is not changed if the identical values are retrieved from these vectors. The relative order between two potential solutions  $M_1$  and  $M_2$  is then characterized by the elements of  $M_1 \Delta M_2$  (the symmetric difference of both sets). Suppose that  $M_1 \Delta M_2 = \{\varphi_1, \dots, \varphi_k\}$ . The score of each formula  $\varphi_i$  in the first solution  $M_1$  is  $g_P^+(\varphi_i)$ . Suppose that  $g_P^+(\varphi_1) \geq \dots \geq g_P^+(\varphi_k)$ .  $M_2$  contains  $\neg\varphi_i$ , of score  $g_P^+(\neg\varphi_i) = n - g_P^+(\varphi_i)$ , thus  $g_P^+(\neg\varphi_k) \geq \dots \geq g_P^+(\neg\varphi_1)$ . Thus the score of  $\varphi_i$  is at rank  $j$  in the ordered score vector associated with  $M_1$  iff the score of  $\varphi_i$  is at rank  $m - j$  in the ordered score vector associated with  $M_2$ .

When we compare  $M_1$  and  $M_2$  using  $leximax$ , the scores are sorted in descending order. Suppose for example that  $M_1 >_{leximax} M_2$ . Then there is  $i \in [1, k]$  s.t.  $g_P^+(\varphi_1) = g_P^+(\neg\varphi_k) \dots g_P^+(\varphi_{i-1}) = g_P^+(\neg\varphi_{k-i+1})$  and  $g_P^+(\varphi_i) > g_P^+(\neg\varphi_{k-i})$ . When we compare  $M_1$  and  $M_2$  using  $leximin$  the scores are sorted in ascending order. For  $M_1$  the vector of scores we have to consider is  $(g_P^+(\varphi_k), \dots, g_P^+(\varphi_1))$  and for  $M_2$  it is  $(g_P^+(\neg\varphi_1), \dots, g_P^+(\neg\varphi_k))$ . Since for  $l \in [1, i - 1]$ ,  $g_P^+(\varphi_l) = g_P^+(\neg\varphi_{k-l+1})$ , we have  $g_P^+(\neg\varphi_l) = g_P^+(\varphi_{k-l+1})$ . So  $g_P^+(\neg\varphi_1) = g_P^+(\varphi_k) \dots g_P^+(\neg\varphi_{i-1}) = g_P^+(\varphi_{k-i+1})$  and  $g_P^+(\neg\varphi_i) < g_P^+(\varphi_{k-i})$ :  $M_2 <_{leximin} M_1$ : the order between  $M_1$  and  $M_2$  is the same one, whatever  $leximax$  or  $leximin$  is considered.  $\square$

This result is surprising, since  $leximax$  and  $leximin$  are quite different operators: with  $leximin$ , more importance is given to the minimal values, whereas with  $leximax$ , more importance is given to the maximal values. Thus, for irresolute profiles  $\delta_{RM_{leximax}}$  and  $\delta_{RM_{leximin}}$  lead usually to different results. This illustrates again the fact that the "no abstention" hypothesis is not a harmless technical simplification; indeed, it may change the results drastically in the sense that methods which differ in general may coincide when only resolute judgment sets are considered.

Some additional results are obtained by focusing on specific ranked majority methods:

**PROPOSITION 8.** •  $\delta_{RM_{\Sigma}}$  satisfies universal domain, anonymity, neutrality, majority preservation, collective rationality and R-swap optimality. It does not satisfy unanimity nor swap optimality.

•  $\delta_{RM_{Maj}}$  satisfies universal domain, anonymity, neutrality, majority preservation, collective rationality. It does not satisfy unanimity nor R-swap optimality.

•  $\delta_{RM_{leximin}}$  satisfies universal domain, anonymity, neutrality, majority preservation, collective rationality, unanimity and R-swap optimality. It does not satisfy swap optimality.

**PROOF.** We give only some proofs, due to space limitation.

•  $\delta_{RM_{\Sigma}}$  does not satisfy unanimity.

Consider the following agenda:  $a, \varphi_1 = a \rightarrow \neg b \vee \neg c, b, c, \varphi_2 = a \rightarrow \neg d \vee \neg e, d, e, \varphi_3 = a \rightarrow \neg f \vee \neg g, f, g, \varphi_4 = a \rightarrow \neg h \vee \neg i, h, i$  and three voters:

$a$	$\varphi_1$	$b$	$c$	$\varphi_2$	$d$	$e$	$\varphi_3$	$f$	$g$	$\varphi_4$	$h$	$i$
1	1	1	0	1	1	0	1	1	0	1	1	0
1	1	0	1	1	0	1	1	0	1	1	0	1
1	0	1	1	0	1	1	0	1	1	0	1	1

Consider any potential solution  $M_a$  containing  $a$ . As  $a$  is in  $M_a$ , one of the three formulae  $a \rightarrow \neg b \vee \neg c, b$  or  $c$  has to be withdrawn from  $M_a$ , one of  $a \rightarrow \neg d \vee \neg e, d, e$  also, one of  $a \rightarrow \neg f \vee \neg g, f, g$  and one of  $a \rightarrow \neg h \vee \neg i, h, i$ . The support of  $M_a$  is  $s(M_a) = (3, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1)$ , and its aggregated score is 23. Consider now any potential solution  $M_{\neg a}$  containing  $\neg a$ . Then all three formulas  $a \rightarrow \neg b \vee \neg c, b$  and  $c$  can be kept,  $a \rightarrow \neg d \vee \neg e, d, e$  also, the same for  $a \rightarrow \neg f \vee \neg g, f, g$  and  $a \rightarrow \neg h \vee \neg i, h, i$ . So the support of  $M_{\neg a}$  is  $s(M_{\neg a}) = (2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 0)$  and its aggregated score is 24.

Accordingly  $M_{\neg a}$  is selected by  $\delta_{RM_{\Sigma}}$ , and unanimity is not satisfied.

•  $\delta_{RM_{\Sigma}}$  does not satisfy swap optimality.

We consider the following example:

$(a \leftrightarrow b)$	1	1	0	0	0	*	*
$a$	*	*	*	1	0	1	*
$b$	*	*	*	0	1	*	1

The score of the potential solution  $M_1 = \{a, b, a \leftrightarrow b\}$  is  $s(M_1) = (2, 2, 2)$  and  $\Sigma(s(M_1)) = 6$ . It is easy to check that  $\Sigma(s(M_2)) = 4$ , where  $M_2 = \{\neg a, \neg b, a \leftrightarrow b\}$ , and  $\Sigma(s(M_3)) = \Sigma(s(M_4)) = 6$  where  $M_3 = \{a, \neg b, \neg(a \leftrightarrow b)\}$  and  $M_4 = \{\neg a, b, \neg(a \leftrightarrow b)\}$  is  $(3, 2, 1)$ .  $\gamma_{RM_{\Sigma}}$  selects  $M_1, M_3$  and  $M_4$ . However,  $M_1$  contains formulae  $a$  and  $a \leftrightarrow b$  whose supports are lower than the support of  $\neg a$ :  $\max(g_P^+(\neg a), g_P^+(\neg(a \leftrightarrow b))) > \max(g_P^+(a), g_P^+(a \leftrightarrow b))$ .  $\gamma_{RM_{\Sigma}}$  does not satisfy swap optimality.

•  $\delta_{RM_{leximin}}$  satisfies unanimity.

Suppose that a formula  $\varphi$  is supported by all agents, and let  $M$  be a potential solution containing all unanimously supported formulae ( $M$  exists because each agent gives a consistent judgment set). Let  $M'$  be another potential solution containing  $\neg\varphi$ . The score of  $\neg\varphi$  is 0, since each agent supports  $\varphi$ . Then the score of  $M'$  contains at least one 0, whereas the score of  $M$  does not (since  $M$  contains all unanimously supported formulae). As a consequence,  $s(M') <_{leximin} s(M)$ , and all unanimously supported formulae are in all elements of  $\delta_{RM_{leximin}}(P)$ .  $\square$

## 5. CHOOSING RELIABLE SOLUTIONS

Besides rationality properties, another criterion for selecting a judgment aggregation method is its capacity to truth tracking. This capacity is assessed as the probability to point out the "true" collective judgment set, assuming that the individuals are reliable, i.e., each agent is more likely to vote in favour of an issue which is true than abstaining or rejecting the issue. More formally, suppose that there is a true judgment set  $\gamma^*$  on  $X$ , i.e.,  $\gamma^*(\varphi_k) = 1$  when  $\varphi_k$  is true, and  $\gamma^*(\varphi_k) = 0$  otherwise. An agent  $i$  is said to be *reliable* if for any issue  $\varphi_k \in X$ ,  $P(\gamma_i(\varphi_k) = \gamma^*(\varphi_k)) > 0.5$ .

As usual for truth tracking analysis [4, 14, 7], a number of assumptions has to be made. Thus the  $n$  agents are supposed to make their decisions independently of others (assumption of individuals independence) and independently of decisions made on other issues (assumption of issues independence). Furthermore, for the sake of simplicity, an homogeneity assumption is often made, stating that the probability of an agent to make the right choice on an issue is the same one, whatever the agent and the issue, i.e.,  $\forall i \in \{1, \dots, n\}, \forall k \in \{1, \dots, m\}, P(\gamma_i(\varphi_k) = \gamma^*(\varphi_k)) = p$ .

These hypotheses are the same ones as those considered in Condorcet's Jury Theorem [4, 14] for voting. This theorem states that for a given issue  $\varphi_k$ , if the individuals are reliable, homogeneous and independent, then the most probable judgment on  $\varphi_k$  is the one supported by a majority of individuals; and if the size of the profile tends to infinity, then the probability to determine the "true" judgment on the issue tends to 1. The Jury Theorem is actually one of the main justifications of our ranked majority methods; indeed, the assumptions underlying this theorem imply that the more supported an opinion on an issue the more likely this opinion.

Now that this is stated, our purpose is to determine whether the truth tracking criterion can be used to discriminate further the ranked majority methods. In order to answer this question, one considers it as a maximum likelihood estimation problem: the aim is to determine the judgment aggregation methods which best explain the observed individual judgments, under the above assumptions.

**PROPOSITION 9.** *Under the assumptions of individuals independence, homogeneity, reliability and issues independence, the judgment aggregation method giving the maximum likelihood estimation of the profile  $P$  on  $X$  is  $\delta_{RM\Sigma}$ .*

**PROOF.** In order to show this result, we prove that under the assumptions of individuals independence, homogeneity, and issues independence, the probability that the profile  $P = (\gamma_1, \dots, \gamma_n)$  is observed when the potential solution  $M = (\alpha_1, \dots, \alpha_m)$  is the true one  $M^*$  is (with  $c(M) = \sum_{i=1}^m g_P^+(\alpha_i)$ ):

$$\begin{aligned} P(P = (\gamma_1, \dots, \gamma_n) \mid M = M^*) &= p^{c(M)} (1-p)^{n \cdot m - c(M)}. \\ P(P = (\gamma_1, \dots, \gamma_n) \mid M = M^*) &= P(P_1^1 = \gamma_1(\varphi_1) \cap \\ P_1^2 = \gamma_1(\varphi_2) \cap \dots \cap P_1^m = \gamma_1(\varphi_m) \cap P_2^1 &= \gamma_2(\varphi_1) \cap \dots \cap \\ P_n^m = \gamma_n(\varphi_m) \mid M = M^*). \end{aligned}$$

Since  $\cap$  is commutative and associative:

$$\begin{aligned} P(P = (\gamma_1, \dots, \gamma_n) \mid M = M^*) &= P(P_1^1 = \gamma_1(\varphi_1) \cap \\ P_2^1 = \gamma_2(\varphi_1) \cap \dots \cap P_n^1 &= \gamma_n(\varphi_1) \cap P_1^2 = \gamma_1(\varphi_2) \cap \dots \cap \\ P_n^m = \gamma_n(\varphi_m) \mid M = M^*). \end{aligned}$$

Using the independence assumption between formulae, we get:

$$\begin{aligned} P(P = (\gamma_1, \dots, \gamma_n) \mid M = M^*) &= P(P_1^1 = \gamma_1(\varphi_1) \cap \\ P_2^1 = \gamma_2(\varphi_1) \cap \dots \cap P_n^1 &= \gamma_n(\varphi_1) \mid M = M^*) \cdot \\ P(P_1^2 = \gamma_1(\varphi_2) \cap \dots \cap P_n^2 &= \gamma_n(\varphi_2) \mid M = M^*) \cdot \dots \cdot \\ P(P_m^1 = \gamma_1(\varphi_m) \cap \dots \cap P_n^m &= \gamma_n(\varphi_m) \mid M = M^*). \end{aligned}$$

Taking advantage of the independence assumption between agents, since exactly  $c(\alpha_1)$  agents among the  $n$  agents gives the right value (probability  $p > 0.5$  because of the reliability assumption) and the  $n - c(\alpha_1)$  remaining ones do not (probability  $1 - p$ ), we get:  $P(P_1^1 = \gamma_1(\varphi_1) \cap P_2^1 = \gamma_2(\varphi_1) \cap \dots \cap P_n^1 = \gamma_n(\varphi_1) \mid M = M^*) = p^{c(\alpha_1)} (1-p)^{n-c(\alpha_1)}$ .

And then:

$$P(P = (\gamma_1, \dots, \gamma_n) \mid M = M^*) = p^{c(\alpha_1)} (1-p)^{n-c(\alpha_1)} \cdot \dots \cdot p^{c(\alpha_m)} (1-p)^{n-c(\alpha_m)} = p^{c(M)} (1-p)^{n \cdot m - c(M)}.$$

Since the  $\delta_{RM\Sigma}$  method selects the potential solutions  $M$  which maximize the score  $c(M) = \sum_{i=1}^m g_P^+(\alpha_i)$ , it gives the maximum likelihood estimation of the profile  $P$ .  $\square$

Thus  $\delta_{RM\Sigma}$  is the best method among judgment aggregation methods (and not only among ranked majority methods) for identifying the true solution, under the assumptions made.

As we mentioned above, the assumptions above are quite standard ones when voting methods are considered [25]. The most debatable one is the assumption of issues independence, which is not satisfied in many cases:

**EXAMPLE 3.** *Consider two formulae,  $\varphi_1 = a \wedge b$  and  $\varphi_2 = \neg a$ . If an agent votes for  $\varphi_1$ , we know that she does not support  $\varphi_2$  (since  $\varphi_2$  is inconsistent with  $\varphi_1$ ). Hence  $P(g_P^+(\varphi_1) = k_1 \wedge g_P^+(\varphi_2) = k_2)$  is different from  $P(g_P^+(\varphi_1) = k_1) \cdot P(g_P^+(\varphi_2) = k_2)$ .*

In this case, the assumption of issues independence is not satisfied<sup>8</sup>, so  $\delta_{RM\Sigma}$  is no longer justified as the best choice for a judgment aggregation method. Accordingly, we now get rid of the assumption of issues independence while still considering the problem of discriminating the ranked majority methods thanks to maximum likelihood estimations. A good point is that, even when the assumption of issues independence does not hold, a given issue is as likely as the number of votes it receives is high. This means that truth tracking can be done on a issue-by-issue basis, but then we have to make decisions based on a vector of more or less supported (hence more or less likely) formulae.

Intuitively, three very natural policies consists then in favoring judgment sets containing as much as possible issues supported by a majority, or in favoring judgment sets containing as much as possible most likely issues, or finally in favoring judgment sets containing as few as possible less likely issues. Formally:

**DEFINITION 10.** *A judgment aggregation method  $\delta$  satisfies the **Best/Card/Worst** policy iff, for any profile  $P$ , all the potential solutions of  $\delta(P)$  are preferred to the other potential solutions w.r.t. **Best/Card/Worst** criterion.*

**Card** *A potential solution  $M$  is preferred to a potential solution  $M'$  w.r.t. the **Card** criterion iff  $|\{\alpha_k \in M \mid g_P^+(\alpha_k) > n/2\}| > |\{\alpha_k \in M' \mid g_P^+(\alpha_k) > n/2\}|$ .*

**Best** *A potential solution  $M$  is preferred to a potential solution  $M'$  w.r.t. the **Best** criterion iff  $\exists \alpha_k \in M$  s.t.  $\forall \alpha_l \in M', g_P^+(\alpha_k) > g_P^+(\alpha_l)$ .*

**Worst** *A potential solution  $M$  is preferred to a potential solution  $M'$  w.r.t. the **Worst** criterion iff  $\exists \alpha_l \in M'$  s.t.  $\forall \alpha_k \in M, g_P^+(\alpha_k) > g_P^+(\alpha_l)$ .*

The preferred judgment sets w.r.t. the three criteria do not coincide in the general case. According to the **Card** criterion the judgment sets containing a maximal number of issues supported by a majority of agents, hence a maximal number of likely issues, are selected. However, this does not prevent from selecting judgment sets containing issues having a very low support, and avoiding issues which are very supported. Worse, when the number of abstentions is high, it can be the case that no issue at all is supported by a majority of agents. In this case, **Card** is not discriminative enough. Alternative choices are given by the **Best** criterion and the **Worst** criterion, which evaluates respectively the likelihood of a judgment set as the best (worst) support of an issue in it. Thus **Best** (resp. **Worst**) corresponds to an optimistic (resp. pessimistic) evaluation

<sup>8</sup>Note that a similar assumption of issues independence is made when maximum likelihood is used for defining the best preference order by aggregation methods in voting theory, despite the fact that similar problematic cases can be pointed out (see e.g. [25]).

of a judgment set. The following proposition shows that for each policy among **Best/Card/Worst**, there is a ranked majority correspondence which satisfies it:

PROPOSITION 10. •  $\delta_{RM_{Maj}}$  satisfies the **Card** policy.

•  $\delta_{RM_{leximax}}$  satisfies the **Best** policy.

•  $\delta_{RM_{leximin}}$  satisfies the **Worst** policy.

## 6. CONCLUSION

This paper gathers a number of results concerning the judgment aggregation problem. The main contributions are:

- A discussion of the rationality properties of judgment aggregation, especially a criticism of the systematicity property, and the introduction of a new property (swap optimality).
- An impossibility theorem.
- A new family of judgment aggregation methods, the support-based aggregation correspondences, for which the selection of consistent collective judgment sets is based only on the support obtained by their elements. We specifically focus on the subset of ranked majority methods, and prove them to satisfy most of the expected rationality properties.
- A truth-tracking justification of some specific operators of this family.

This work opens several perspectives. One of them consists in considering other rationality postulates for judgment aggregation (as responsiveness and monotonicity, considered in [9]) and then to identify the maximal sets of properties which can be jointly satisfied, to go from impossibility theorems to "possibility theorems".

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